

Resonant bands and local system cohomology groups for real line arrangements

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Abstract

We give a new algorithm computing local system cohomology groups for complexified real line arrangements. Using it, we obtain several conditions for the first local system cohomology to vanish and to be at most one-dimensional, which generalize a result by Cohen-Dimca-Orlik. The conditions are described in terms of discrete geometric structures of real figures. The proof is based on a recent study on minimal cell structures. We also compute the characteristic variety of the deleted B_3 -arrangement.

1 Introduction

In the theory of hyperplane arrangements, one of the central problems is to what extent topological invariants of the complements are determined combinatorially. For example, the cohomology ring is combinatorially determined (Orlik and Solomon), while the fundamental group is not (Rybnikov). Between these two cases, local system cohomology groups and monodromy eigenspaces of Milnor fibers recently received a considerable amount of attention.

There are several ways to compute local system cohomology groups, especially for line arrangements. In this paper, we use the twisted minimal complex in [7, 8, 9]. Since the complex is described in terms of adjacency relations of chambers, we can employ discrete-geometric arguments to the computation of local system cohomology groups. By using combinatorial arguments, we obtain several conditions on rank-one local systems for the first

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cohomology to vanish, also a condition for the first cohomology is at most one-dimensional.

The paper is organized as follows. In §2 we fix some notation and recall the twisted minimal complex. §3 is the main section of the paper. First, in §3.1 we introduce discrete geometric notions, the so-called *\mathcal{L} -resonant band* and the *standing wave* on this band. These notions were first introduced in [11] in the study of eigenspaces of monodromy action on the Milnor fibers. They are also useful for the computation of local system cohomology groups. We give an algorithm that computes local system cohomology group in terms of resonant bands in §3.2. Then in §3.3 and §3.4, we give a few upper bounds of $\dim H^1$. Finally in §4, we apply our algorithm to the deleted B_3 -arrangement and compute the characteristic variety.

2 Preliminaries

2.1 Line arrangements and local systems

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine line arrangement in \mathbb{R}^2 with the defining equation $Q_{\mathcal{A}}(x, y) = \prod_{i=1}^n \alpha_i$, where α_i is a defining linear equation for H_i . In this paper, we assume that not all lines are parallel (or equivalently, \mathcal{A} has at least one intersection). The coning $c\mathcal{A}$ of \mathcal{A} is an arrangement of $n + 1$ planes in \mathbb{R}^3 defined by the equation $Q_{c\mathcal{A}}(x, y, z) = z^{n+1}Q(\frac{x}{z}, \frac{y}{z})$. The line $\{z = 0\} \in c\mathcal{A}$ is called the line at infinity and denoted by H_{∞} . The space $M(\mathcal{A}) = \mathbb{C}^2 \setminus \{Q_{\mathcal{A}} = 0\} = \mathbb{P}_{\mathbb{C}}^2 \setminus \{Q_{c\mathcal{A}} = 0\}$ is called the complexified complement. In this article, \mathcal{A} always denote a line arrangement in \mathbb{R}^2 and $c\mathcal{A}$ denotes that in \mathbb{RP}^2 .

A rank-one local system \mathcal{L} is determined by a homomorphism

$$\pi_1(M(\mathcal{A}), *) \longrightarrow \mathbb{C}^*.$$

Since the right-hand side is Abelian, the map can be lift to $H_1(M(\mathcal{A})) \longrightarrow \mathbb{C}^*$. Recall that $H_1(M(\mathcal{A}))$ is a free Abelian group generated by meridian loops around $H_i \in \mathcal{A}$. Hence \mathcal{L} is determined by the point $(q_1, \dots, q_n) \in (\mathbb{C}^*)^n$ of the character torus, where $q_i \in \mathbb{C}^*$ is a monodromy along the meridian loop around the line H_i . The monodromy around the line at infinity H_{∞} is automatically determined as

$$q_{\infty} = (q_1 q_2 \cdots q_n)^{-1}.$$

Let us denote by $\mathcal{L}_{\mathbf{q}}$ the local system determined by $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{C}^n$. Let $X \subset \mathbb{RP}^2$ be a subset (e.g. a point or a line). Denote $\mathcal{A}_X = \{H \in c\mathcal{A} \mid H \supset X\}$ and $q_X = \prod_{H \in \mathcal{A}_X} q_H$. We fix some terminology.

Definition 2.1. (1) A line $H \in c\mathcal{A}$ is said to be *resonant* if $q_H = 1$.

(2) A point $X \in \mathbb{RP}^2$ is called a *multiple point* if at least three lines in $c\mathcal{A}$ are passing through X .

(3) A multiple point X is called a *resonant point* if $q_X = 1$.

We also use notations $q_{ijk} := q_i q_j q_k$, $q_{ijk}^{1/2} := q_i^{1/2} q_j^{1/2} q_k^{1/2}$.

2.2 Twisted minimal cochain complexes

In this section, we recall the construction of the twisted minimal cochain complex from [7, 8, 9].

A connected component of $\mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\text{ch}(\mathcal{A})$. A chamber $C \in \text{ch}(\mathcal{A})$ is called bounded (resp. unbounded) if the area is finite (resp. infinite). For an unbounded chamber $U \in \text{ch}(\mathcal{A})$, the opposite unbounded chamber is denoted by U^\vee (see [9, Definition 2.1] for the definition; see also Figure 1 below).

Let \mathcal{F} be a generic flag in \mathbb{R}^2

$$\mathcal{F} : \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 = \mathbb{R}^2,$$

where \mathcal{F}^k is a generic k -dimensional affine subspace.

Definition 2.2. For $k = 0, 1, 2$, define the subset $\text{ch}_{\mathcal{F}}^k(\mathcal{A}) \subset \text{ch}(\mathcal{A})$ by

$$\text{ch}_{\mathcal{F}}^k(\mathcal{A}) := \{C \in \text{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^k \neq \emptyset, C \cap \mathcal{F}^{k-1} = \emptyset\}.$$

The set of chambers decomposes into a disjoint union as $\text{ch}(\mathcal{A}) = \text{ch}_{\mathcal{F}}^0(\mathcal{A}) \sqcup \text{ch}_{\mathcal{F}}^1(\mathcal{A}) \sqcup \text{ch}_{\mathcal{F}}^2(\mathcal{A})$. The cardinality of $\text{ch}_{\mathcal{F}}^k(\mathcal{A})$ is equal to $b_k(\mathbf{M}(\mathcal{A}))$ for $k = 0, 1, 2$.

We further assume that the generic flag \mathcal{F} satisfies the following conditions:

- \mathcal{F}^1 does not separate intersections of \mathcal{A} ,
- \mathcal{F}^0 does not separate n -points $\mathcal{A} \cap \mathcal{F}^1$.

Then we can choose coordinates x_1, x_2 so that \mathcal{F}^0 is the origin $(0, 0)$, \mathcal{F}^1 is given by $x_2 = 0$, all intersections of \mathcal{A} are contained in the upper-half plane $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ and $\mathcal{A} \cap \mathcal{F}^1$ is contained in the half-line $\{(x_1, 0) \mid x_1 > 0\}$.

We set $H_i \cap \mathcal{F}^1$ to have coordinates $(a_i, 0)$. By changing the numbering of the lines and the signs of the defining equation α_i of $H_i \in \mathcal{A}$ we may assume that

- $0 < a_1 < a_2 < \dots < a_n$,
- the origin \mathcal{F}^0 is contained in the negative half-plane $H_i^- = \{\alpha_i < 0\}$.

We set $\text{ch}_0^{\mathcal{F}}(\mathcal{A}) = \{U_0\}$ and $\text{ch}_1^{\mathcal{F}}(\mathcal{A}) = \{U_1, \dots, U_{n-1}, U_0^\vee\}$ so that $U_p \cap \mathcal{F}^1$ is equal to the interval (a_p, a_{p+1}) for $p = 1, \dots, n-1$. It is easily seen that the chambers U_0, U_1, \dots, U_{n-1} and U_0^\vee have the following expression:

$$\begin{aligned} U_0 &= \bigcap_{i=1}^n \{\alpha_i < 0\}, \\ U_p &= \bigcap_{i=1}^p \{\alpha_i > 0\} \cap \bigcap_{i=p+1}^n \{\alpha_i < 0\}, \quad (p = 1, \dots, n-1), \\ U_0^\vee &= \bigcap_{i=1}^n \{\alpha_i > 0\}. \end{aligned} \tag{1}$$

The notations introduced to this point are illustrated in Figure 1.

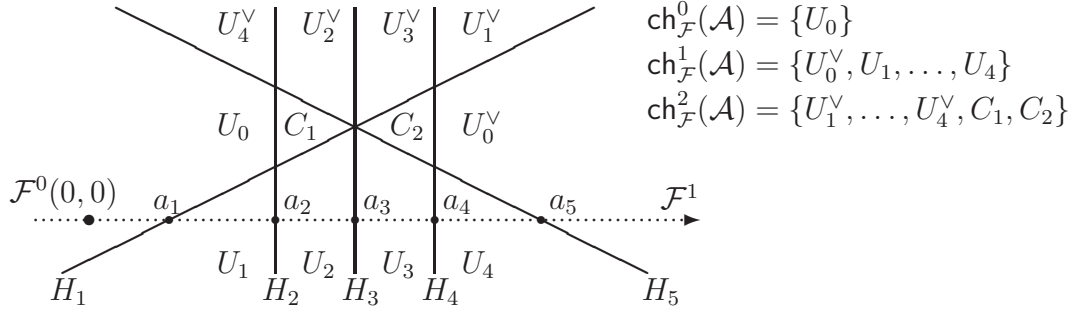


Figure 1: Numbering of lines and chambers.

Let \mathcal{L} be a complex rank-one local system on $\mathbf{M}(\mathcal{A})$. The local system \mathcal{L} is determined by non-zero complex numbers (monodromy around H_i) $q_i \in \mathbb{C}^*$, $i = 1, \dots, n$. Fix a square root $q_i^{1/2} \in \mathbb{C}^*$ for each i .

Definition 2.3. (1) For $C, C' \in \text{ch}(\mathcal{A})$, let us denote by $\text{Sep}(C, C')$ the set of lines $H_i \in \mathcal{A}$ which separate C and C' .

(2) Define the complex number $\Delta(C, C') \in \mathbb{C}$ by

$$\Delta(C, C') := \prod_{H_i \in \text{Sep}(C, C')} q_i^{1/2} - \prod_{H_i \in \text{Sep}(C, C')} q_i^{-1/2}.$$

Now we construct the cochain complex $(\mathbb{C}[\text{ch}_{\mathcal{F}}^\bullet(\mathcal{A})], d_{\mathcal{L}})$.

(i) The map $d_{\mathcal{L}} : \mathbb{C}[\text{ch}_{\mathcal{F}}^0(\mathcal{A})] \longrightarrow \mathbb{C}[\text{ch}_{\mathcal{F}}^1(\mathcal{A})]$ is defined by

$$d_{\mathcal{L}}([U_0]) = \Delta(U_0, U_0^\vee)[U_0^\vee] + \sum_{p=1}^{n-1} \Delta(U_0, U_p)[U_p].$$

(ii) $d_{\mathcal{L}} : \mathbb{C}[\text{ch}_{\mathcal{F}}^1(\mathcal{A})] \longrightarrow \mathbb{C}[\text{ch}_{\mathcal{F}}^2(\mathcal{A})]$ is defined by

$$d_{\mathcal{L}}([U_p]) = - \sum_{\substack{C \in \text{ch}_{\mathcal{F}}^2(\mathcal{A}) \\ \alpha_p(C) > 0 \\ \alpha_{p+1}(C) < 0}} \Delta(U_p, C)[C] + \sum_{\substack{C \in \text{ch}_{\mathcal{F}}^2(\mathcal{A}) \\ \alpha_p(C) < 0 \\ \alpha_{p+1}(C) > 0}} \Delta(U_p, C)[C], \quad (\text{for } p = 1, \dots, n-1),$$

$$d_{\mathcal{L}}([U_0^\vee]) = - \sum_{\alpha_n(C) > 0} \Delta(U_0^\vee, C)[C].$$

Example 2.4. Let $\mathcal{A} = \{H_1, \dots, H_5\}$, and let the flag \mathcal{F} be as in Figure 1. Then

$$d_{\mathcal{L}}([U_0]) = ([U_1], [U_2], [U_3], [U_4], [U_0^\vee]) \begin{pmatrix} q_1^{1/2} - q_1^{-1/2} \\ q_{12}^{1/2} - q_{12}^{-1/2} \\ q_{123}^{1/2} - q_{123}^{-1/2} \\ q_{1234}^{1/2} - q_{1234}^{-1/2} \\ q_{12345}^{1/2} - q_{12345}^{-1/2} \end{pmatrix},$$

$$d_{\mathcal{L}}([U_1], [U_2], [U_3], [U_4], [U_0^\vee]) = ([U_1^\vee], [U_2^\vee], [U_3^\vee], [U_4^\vee], [C_1], [C_2])$$

$$\times \begin{pmatrix} q_{12345}^{1/2} - q_{12345}^{-1/2} & 0 & 0 & 0 & -(q_1^{1/2} - q_1^{-1/2}) \\ q_{125}^{1/2} - q_{125}^{-1/2} & -(q_{15}^{1/2} - q_{15}^{-1/2}) & 0 & q_{1345}^{1/2} - q_{1345}^{-1/2} & -(q_{134}^{1/2} - q_{134}^{-1/2}) \\ q_{1235}^{1/2} - q_{1235}^{-1/2} & 0 & -(q_{15}^{1/2} - q_{15}^{-1/2}) & q_{145}^{1/2} - q_{145}^{-1/2} & -(q_{14}^{1/2} - q_{14}^{-1/2}) \\ 0 & 0 & 0 & q_{12345}^{1/2} - q_{12345}^{-1/2} & -(q_{1234}^{1/2} - q_{1234}^{-1/2}) \\ q_{12}^{1/2} - q_{12}^{-1/2} & -(q_1^{1/2} - q_1^{-1/2}) & 0 & 0 & 0 \\ 0 & 0 & -(q_5^{1/2} - q_5^{-1/2}) & q_{45}^{1/2} - q_{45}^{-1/2} & -(q_4^{1/2} - q_4^{-1/2}) \end{pmatrix}.$$

Theorem 2.5. Under the above notation, $(\mathbb{C}[\text{ch}_{\mathcal{F}}^\bullet(\mathcal{A})], d_{\mathcal{L}})$ is a cochain complex and

$$H^k(\mathbb{C}[\text{ch}_{\mathcal{F}}^\bullet(\mathcal{A})], d_{\mathcal{L}}) \simeq H^k(M(\mathcal{A}), \mathcal{L}).$$

See [7, 8, 9] for details.

3 Main result

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine lines in \mathbb{R}^2 . Let \mathcal{F} be a generic flag as in §2.2. We will see that if $q_\infty \neq 1$, then we obtain a simpler algorithm computing $H^1(M(\mathcal{A}), \mathcal{L})$.

3.1 Resonant bands and standing waves

Definition 3.1. A *band* B is a region bounded by a pair of consecutive parallel lines H_i and H_{i+1} .

Each band B includes two unbounded chambers $U_1(B), U_2(B) \in \text{ch}(\mathcal{A})$. By definition, $U_1(B)$ and $U_2(B)$ are opposite each other, $U_1(B)^\vee = U_2(B)$ and $U_2(B)^\vee = U_1(B)$.

Definition 3.2. A band B is called \mathcal{L} -resonant if

$$\Delta(U_1(B), U_2(B)) = 0.$$

We denote the set of all \mathcal{L} -resonant bands by $\text{RB}_{\mathcal{L}}(\mathcal{A})$.

Let us denote by \overline{B} the closure of B in the real projective plane \mathbb{RP}^2 . Then the intersection $X(B) := \overline{B} \cap H_\infty$ is a single point. Each line $H \in \mathcal{A} \cup \{H_\infty\}$ either passes through $X(B)$ or separates $U_1(B)$ and $U_2(B)$.

Proposition 3.3. A band B is \mathcal{L} -resonant if and only if $q_{X(B)} = 1$. The directions of \mathcal{L} -resonant bands are one-to-one correspondence with points $X \in H_\infty$ such that $q_X = 1$.

Proof. By the above remark, $c\mathcal{A}$ is decomposed as

$$c\mathcal{A} = \text{Sep}(U_1(B), U_2(B)) \sqcup (c\mathcal{A})_{X(B)},$$

and we have

$$q_{X(B)} \times \prod_{H_i \in \text{Sep}(U_1, U_2)} q_i = \prod_{H_i \in c\mathcal{A}} q_i = 1.$$

Hence $q_{X(B)} = 1$ if and only if $\prod_{H_i \in \text{Sep}(U_1, U_2)} q_i = 1$, which is equivalent to $\Delta(U_1(B), U_2(B)) = 0$. \square

To an \mathcal{L} -resonant band $B \in \text{RB}_{\mathcal{L}}(\mathcal{A})$ we can associate a *standing wave* $\nabla_i(B) \in \mathbb{C}[\text{ch}(\mathcal{A})]$ ($i = 1, 2$) on the band B as follows:

$$\nabla_i(B) = \sum_{\substack{C \in \text{ch}(\mathcal{A}), \\ C \subset B}} \Delta(U_i(B), C) \cdot [C]. \quad (2)$$

Proposition 3.4. Let B be a \mathcal{L} -resonant band. Then $\nabla_2(B) = \pm \nabla_1(B)$.

Proof. Since B is \mathcal{L} -resonant,

$$\varepsilon := q_{X(B)}^{1/2} = \prod_{H_i \in (c\mathcal{A})_{X(B)}} q_i^{1/2}$$

is either $+1$ or -1 . Let $C \subset B$ be a chamber contained in B . Then $c\mathcal{A}$ is decomposed as $c\mathcal{A} = \text{Sep}(U_1(B), C) \sqcup \text{Sep}(U_2(B), C) \sqcup (c\mathcal{A})_{X(B)}$, and so we have

$$\prod_{H_i \in \text{Sep}(U_1, C)} q_i^{1/2} \times \prod_{H_i \in \text{Sep}(U_2, C)} q_i^{1/2} = \varepsilon.$$

Hence

$$\Delta(U_1(B), C) = -\varepsilon \cdot \Delta(U_2(B), C).$$

We conclude that $\nabla_1(B) = -\varepsilon \cdot \nabla_2(B)$. \square

In the remainder of the paper we denote $\nabla(B) := \nabla_1(B)$ for simplicity.

Remark 3.5. To indicate the choice of $U_1(B)$ and $U_2(B)$, we always put the name B of the band in the unbounded chamber $U_1(B)$ in figures.

3.2 Cohomology via resonant bands

The map $B \mapsto \nabla(B)$ can be naturally extended to the linear map

$$\nabla : \mathbb{C}[\text{RB}_{\mathcal{L}}(\mathcal{A})] \longrightarrow \mathbb{C}[\text{ch}(\mathcal{A})]. \quad (3)$$

Theorem 3.6. *Assume that $q_{\infty} \neq 1$. Then the kernel of ∇ is isomorphic to the first cohomology of \mathcal{L} -coefficients, that is,*

$$\text{Ker}(\nabla : \mathbb{C}[\text{RB}_{\mathcal{L}}(\mathcal{A})] \longrightarrow \mathbb{C}[\text{ch}(\mathcal{A})]) \simeq H^1(M(\mathcal{A}), \mathcal{L}).$$

In particular, $\dim H^1(M(\mathcal{A}), \mathcal{L})$ is equal to the number of linear relations among the standing waves $\nabla(B)$, $B \in \text{RB}_{\mathcal{L}}(\mathcal{A})$.

Proof. We consider the first cohomology group $H^1(\mathbb{C}[\text{ch}_{\mathcal{F}}^{\bullet}(\mathcal{A})], d_{\mathcal{L}})$ of the twisted minimal cochain complex. The image $d_{\mathcal{L}} : \mathbb{C}[\text{ch}_{\mathcal{F}}^0(\mathcal{A})] \longrightarrow \mathbb{C}[\text{ch}_{\mathcal{F}}^1(\mathcal{A})]$ is generated by

$$d_{\mathcal{L}}([U_0]) = \sum_{p=1}^{n-1} \Delta(U_0, U_p)[U_p] \pm (q_{\infty}^{1/2} - q_{\infty}^{-1/2})[U_0^{\vee}].$$

(The sign depends on $q_{12\dots n\infty}^{1/2} = \pm 1$.) Since $q_{\infty} \neq 1$, the coefficient of $[U_0^{\vee}]$ in $d_{\mathcal{L}}([U_0])$ is non-zero. Define the subspace V of $\mathbb{C}[\text{ch}_{\mathcal{F}}^1(\mathcal{A})]$ by

$$\begin{aligned} V &= \bigoplus_{p=1}^{n-1} \mathbb{C} \cdot [U_p] \\ &(\simeq \text{Coker}(d_{\mathcal{L}} : \mathbb{C}[\text{ch}_{\mathcal{F}}^0(\mathcal{A})] \longrightarrow \mathbb{C}[\text{ch}_{\mathcal{F}}^1(\mathcal{A})])). \end{aligned} \quad (4)$$

Then $H^1(\mathbb{C}[\text{ch}_{\mathcal{F}}^{\bullet}(\mathcal{A})], d_{\mathcal{L}})$ is isomorphic to $\text{Ker}(d_{\mathcal{L}}|_V : V \longrightarrow \mathbb{C}[\text{ch}_{\mathcal{F}}^2(\mathcal{A})])$. It is sufficient to show that $\text{Ker}(d_{\mathcal{L}}|_V) \simeq \text{Ker} \nabla$, which will be done in several steps. Suppose that $\varphi = \sum_{p=1}^{n-1} c_p \cdot [U_p] \in \text{Ker}(d_{\mathcal{L}}|_V)$.

- (i) If H_i and H_{i+1} are not parallel, then $c_i = 0$.

Note that if $j \neq i$, then the chamber $[U_i^\vee]$ does not appear in $d_{\mathcal{L}}([U_j])$. Thus the coefficient of $[U_i^\vee]$ in

$$d_{\mathcal{L}}(\varphi) = \sum_{p=1}^{n-1} c_p \cdot d_{\mathcal{L}}([U_p])$$

is $c_i \cdot \Delta(U_i, U_i^\vee) = \pm c_i (q_\infty^{1/2} - q_\infty^{-1/2})$. This equals zero if and only if $c_i = 0$.

Now we may assume that $\varphi = \sum_p c_p \cdot [U_p] \in \text{Ker}(d_{\mathcal{L}})$ is a linear combination of $[U_p]$ s such that H_p and H_{p+1} are parallel. Suppose that H_i and H_{i+1} are parallel and denote by B_i the band determined by these lines.

(ii) If B_i is not \mathcal{L} -resonant, then $c_i = 0$.

In this case, $\Delta(U_i, U_i^\vee) \neq 0$. Since φ is a linear combination of $[U_p]$ s with parallel boundaries H_p and H_{p+1} , the term $[U_i^\vee]$ appears only in $d_{\mathcal{L}}([U_i])$, which is equal to $c_i \cdot \Delta(U_i, U_i^\vee)[U_i^\vee]$. Therefore $c_i = 0$.

Finally we may assume that φ is a linear combination of $[U_p]$ s such that the boundaries H_p and H_{p+1} are parallel and the length of the corresponding band B_p is divisible by k . In this case, it is straightforward to check that the maps $d_{\mathcal{L}}$ and ∇ are identical. This completes the proof. \square

Example 3.7. Let $\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\}$ be the line arrangement in Figure 1. There are two bands:

$$\begin{aligned} B_1 &= (\text{The band bounded by } H_2 \text{ and } H_3), \\ B_2 &= (\text{The band bounded by } H_3 \text{ and } H_4). \end{aligned}$$

We set $U_1(B_1) = U_2, U_2(B_1) = U_2^\vee, U_1(B_2) = U_3$ and $U_2(B_2) = U_3^\vee$. The band B_i is $\mathcal{L}_{\mathbf{q}}$ -resonant if and only if $q_{15} = 1$, or equivalently $q_{234\infty} = 1$. Then we have

$$\begin{aligned} \nabla(B_1) &= \Delta(U_2, C_1) \cdot [C_1] = (q_1^{1/2} - q_1^{-1/2}) \cdot [C_1] \\ \nabla(B_2) &= \Delta(U_3, C_2) \cdot [C_2] = (q_5^{1/2} - q_5^{-1/2}) \cdot [C_2] = \pm (q_1^{1/2} - q_1^{-1/2}) \cdot [C_2]. \end{aligned}$$

Hence,

$$\dim H^1(\mathcal{L}_{\mathbf{q}}) = \begin{cases} 2 & q_1 = q_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This example is a special case of Theorem 3.11.

3.3 Vanishing

We describe some corollaries to Theorem 3.6.

Corollary 3.8. *Suppose that $q_\infty \neq 1$. If every multiple point $X \in H_\infty$ satisfies $q_X \neq 1$, then $H^1(M(\mathcal{A}), \mathcal{L}) = 0$.*

Proof. By Proposition 3.3, the assumption is equivalent to $\text{RB}_{\mathcal{L}}(\mathcal{A}) = \emptyset$. Since $\mathbb{C}[\text{RB}_k(\mathcal{A})] = 0$, obviously $\text{Ker}(\nabla : \mathbb{C}[\text{RB}_k(\mathcal{A})] \rightarrow \mathbb{C}[\text{ch}(\mathcal{A})]) = 0$. By Theorem 3.6, $H^1(M(\mathcal{A}), \mathcal{L}) = 0$. \square

Corollary 3.9. *If there exists a non-resonant line $H_i \in c\mathcal{A}$ such that every multiple point on H_i is non-resonant, then $H^1(M(\mathcal{A}), \mathcal{L}) = 0$.*

Remark 3.10. Corollary 3.9 is proved by Cohen, Dimca and Orlik [1] for more general complex arrangement cases. It is also proved for real line arrangements that the assumption of Corollary 3.9 is equivalent to $H^2(M(\mathcal{A}), \mathcal{L})$ being generated by bounded chambers [9].

For real case, we obtain a modified version.

Theorem 3.11. *Suppose that $q_\infty \neq 1$ and H_∞ has a unique resonant multiple point X . Let $\mathcal{B} = c\mathcal{A} \setminus (c\mathcal{A})_X$ be the set of lines which are not passing through X . Then*

$$\dim H^1(M(\mathcal{A}), \mathcal{L}) = \begin{cases} |(c\mathcal{A})_X| - 2, & \text{if } q_H = 1, \forall H \in \mathcal{B} \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Proof. By the assumption, $\text{RB}_{\mathcal{L}}(\mathcal{A}) = \{B_1, \dots, B_m\}$ consists of parallel bands, where $m = |(c\mathcal{A})_X| - 2$. Now the supports of $\nabla(B_1), \dots, \nabla(B_m)$, that is, the set of chambers appearing in each standing wave, are mutually disjoint. Therefore, $\nabla(B_1), \dots, \nabla(B_m)$ are linearly dependent if and only if $\nabla(B_1) = \dots = \nabla(B_m) = 0$, which is equivalent to that $q_H = 1$ for all $H \in \mathcal{B}$. In this case, we have that the space $\text{Ker}(\nabla) = \mathbb{C}[\text{RB}_{\mathcal{L}}(\mathcal{A})]$ is $(|(c\mathcal{A})_X| - 2)$ -dimensional. \square

Corollary 3.12. *Suppose that there exists a line $H_i \in c\mathcal{A}$ such that $q_i \neq 1$ and there is at most one point $X \in H_i$ satisfying $q_X = 1$. Then $\dim H^1(M(\mathcal{A}), \mathcal{L})$ is combinatorially determined.*

Remark 3.13. We do not know whether Theorem 3.11 holds for complex arrangements.

3.4 Upper-bound

Recall that two lines H, H' in the real projective plane \mathbb{RP}^2 divide the space into two regions.

Definition 3.14. Let $c\mathcal{A}$ be a line arrangement in the real projective plane \mathbb{RP}^2 . A pair of non-resonant lines H_i and $H_j \in c\mathcal{A}$ (that is, they satisfy $q_i \neq 1$ and $q_j \neq 1$) is said to be a *sharp pair* if all intersection points of $c\mathcal{A} \setminus \{H_i, H_j\}$ are contained in one of two regions or lie on $H_i \cup H_j$. (In other words, there are no intersection points in one of the two regions determined by H_i and H_j .)

Example 3.15. In Figure 2, the lines H_1 and H_∞ form a sharp pair (since the left half plane of H_1 does not contain intersections).

If there exists a non-resonant line $H \in c\mathcal{A}$ such that H has at most one resonant multiple point, then $\dim H^1(M(\mathcal{A}), \mathcal{L})$ is computed combinatorially (Corollary 3.12). Thus we may assume that every non-resonant line $H \in c\mathcal{A}$ has at least two resonant multiple points.

Theorem 3.16. *Let \mathcal{A} be a line arrangement and \mathcal{L} be a rank-one local system. Assume that every non-resonant line $H \in c\mathcal{A}$ has at least two resonant points. Suppose that the arrangement $c\mathcal{A}$ contains a sharp pair of non-resonant lines. Then:*

- (i) $\dim H^1(M(\mathcal{A}), \mathcal{L}) \leq 1$.
- (ii) *Suppose that the lines $H_1, H_2 \in c\mathcal{A}$ are non-resonant and form a sharp pair. Let $X = H_1 \cap H_2$ be the intersection. If $(c\mathcal{A})_X = \{H_1, H_2\}$ or $q_X \neq 1$, then $H^1(M(\mathcal{A}), \mathcal{L}) = 0$.*

Proof. By $PGL_3(\mathbb{C})$ action, we may assume that the line at infinity H_∞ and $H_1 = \{x = 0\}$ form a sharp pair of non-resonant lines and there is no intersections in the region $\{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ (see Figure 2). The intersection is $X = H_\infty \cap H_1 = \{(0 : 1 : 0)\}$. Let B be a horizontal (that is, non-vertical) band, that is a band which is not passing through the point X . We choose $U_1(B)$ to be the unbounded chamber in B contained in the region $\{x < 0\}$. Denote by C_B the leftmost bounded chamber in B (e.g., in Figure 2, there are four horizontal band B_1, \dots, B_4 . In this case, $C_{B_1} = C_2, C_{B_2} = C_3, C_{B_3} = C_4$ and $C_{B_4} = C_6$).

First consider the case $q_X \neq 1$. Then all \mathcal{L} -resonant bands are horizontal. Let $B \in \text{RB}_{\mathcal{L}}(\mathcal{A})$. Then

$$\nabla(B) = \Delta(U_1(B), C_B) \cdot [C_B] + \dots \quad (6)$$

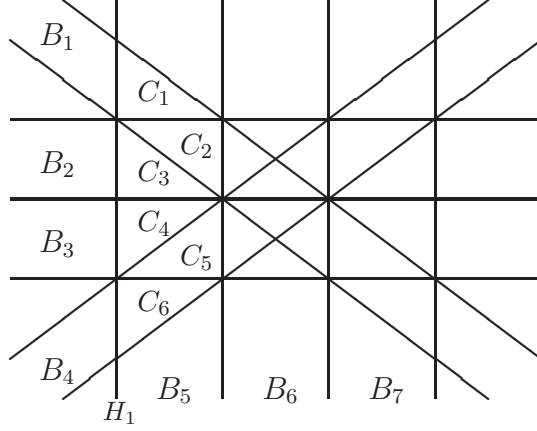


Figure 2: H_1 and H_∞ form a sharp pair

Thus $[C_B]$ has a non-zero coefficient. Note that C_B is contained in the unique \mathcal{L} -resonant band B . Hence the standing waves $\nabla(B), B \in \text{RB}_k(\mathcal{A})$ are linearly independent. Thus (ii) is proved.

Now we assume that $q_X = 1$. In this case, there are vertical \mathcal{L} -resonant bands. Denote by B_{left} the leftmost vertical band (in Figure 2, $B_{\text{left}} = B_5$). Suppose that

$$c_{\text{left}} \cdot B_{\text{left}} + \cdots \in \text{Ker}(\nabla).$$

Let $B \in \text{RB}_{\mathcal{L}}(\mathcal{A})$ be a horizontal \mathcal{L} -resonant band. Then, since C_B is contained in only B and B_{left} , the coefficient c_{left} of B_{left} determines the coefficient of B . The coefficients of other vertical \mathcal{L} -resonant bands are also determined by those of the horizontal bands. Hence $\text{Ker}(\nabla)$ is at most one-dimensional. \square

Corollary 3.17. *Suppose that $\dim H^1(M(\mathcal{A}), \mathcal{L}) \geq 2$. Then for every sharp pair $H, H' \in c\mathcal{A}$ of non-resonant lines, there exist intersections of $c\mathcal{A}$ in both two regions determined by H and H' .*

Example 3.18. Let $\mathcal{A} = \{H_1, \dots, H_8\}$ be the deleted B_3 -arrangement (see §4.1 for the definition) and let $\mathbf{q}_+ = (q_1, \dots, q_8) = (1, -1, -1, 1, 1, -1, 1, -1)$. We compute $H^1(M(\mathcal{A}), \mathcal{L}_{\mathbf{q}_+})$. Since $q_8 \neq 1$, we can apply Theorem 3.6 to Figure 4. There are four resonant bands: $\text{RB}_{\mathcal{L}_{\mathbf{q}_+}} = \{B_1, B_2, B_3, B_4\}$. Set $q_1^{1/2} = q_4^{1/2} = q_5^{1/2} = q_7^{1/2} = 1$ and $q_2^{1/2} = q_3^{1/2} = q_6^{1/2} = q_8^{1/2} = i$. Then we have

$$\nabla(B_1) = \nabla(B_2) = \nabla(B_4) = 2i \cdot C_2, \quad \nabla(B_3) = 2i(C_4 + C_5 + C_6),$$

and so $B_1 - B_2, B_2 - B_3 \in \text{Ker}(\nabla)$ form a basis. We have $\dim H^1(M(\mathcal{A}), \mathcal{L}_{\mathbf{q}_+}) = 2$.

If we set $\mathbf{q}_- = (q_1, \dots, q_8) = (-1, 1, 1, -1, 1, -1, 1, -1)$, similarly we have

$$\nabla(B_1) = \nabla(B_3) = \nabla(B_4) = 2i \cdot C_5, \quad \nabla(B_2) = 2i(C_1 + C_2 + C_3),$$

and $\dim H^1(M(\mathcal{A}), \mathcal{L}_{\mathbf{q}_-}) = 2$.

4 Example: The deleted B_3 -arrangement

Using Theorem 3.6, we can compute the characteristic variety of line arrangements. We apply the following strategy to the deleted B_3 -arrangement.

- (1) Fix a line H .
 - Put H at ∞ and assume that $q_H \neq 1$.
 - Using Theorem 3.6, compute the characteristic variety using the assumption that $q_H \neq 1$.
- (2) Assume that $q_H = 1$ and choose another line H' , go back to (1).

4.1 Deleted B_3 -arrangement

The deleted B_3 -arrangement is an arrangement of 8 lines in \mathbb{RP}^2 defined by the following equations.

$$\begin{aligned}
H_1 : y &= 0 \\
H_2 : y - z &= 0 \\
H_3 : x &= 0 \\
H_4 : x - z &= 0 \\
H_5 : x - y + z &= 0 \\
H_6 : x - y &= 0 \\
H_7 : x - y - z &= 0 \\
H_8 : z &= 0.
\end{aligned} \tag{7}$$

(We use the numbering in [5, Example 10.6]. See Figure 3, 4, 5 and 6 below.)

In the sequel we compute the characteristic variety

$$V_1(M(\mathcal{A})) = \{\mathbf{q} = (q_1, \dots, q_8) \in (\mathbb{C}^*)^8 \mid q_1 q_2 \cdots q_8 = 1, \dim H^1(M(\mathcal{A}), \mathcal{L}_{\mathbf{q}}) \geq 1\} \tag{8}$$

of the deleted B_3 -arrangement (without using computer).

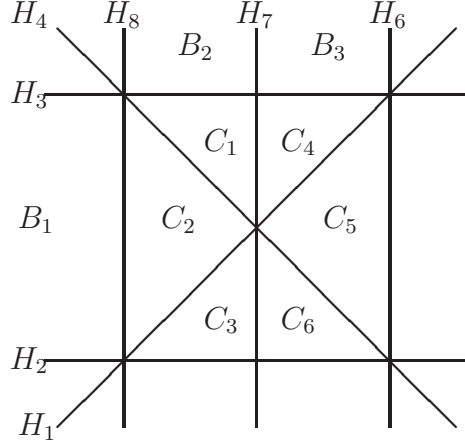


Figure 3: The deleted B_3 -arrangement with line at infinity H_5

4.2 The case $q_5 \neq 1$

First, we consider the case $q_5 \neq 1$. There are two multiple points 235 and 5678 on H_5 (Figure 3).

There four cases.

- (1) $q_{235} \neq 1, q_{5678} \neq 1$. Then by Corollary 3.8, we have $H^1(\mathcal{L}_q) = 0$.
- (2) $q_{235} \neq 1, q_{5678} = 1$. Then by Theorem 3.11, $H^1(\mathcal{L}_q) = 0$ if and only if $q_1 = q_2 = q_3 = q_4 = 1$ (and then $\dim H^1 = 2$). It is corresponding to the component C_{5678} in [5, Example 10.6].
- (3) $q_{235} = 1, q_{5678} \neq 1$. Then by Theorem 3.11, $H^1(\mathcal{L}_q) = 0$ if and only if $q_1 = q_4 = q_6 = q_7 = q_8 = 1$ (and then $\dim H^1 = 1$). It is corresponding to the component C_{235} in [5, Example 10.6].
- (4) $q_{235} = q_{5678} = 1$. This case requires more detailed analysis as follows.

In the case (4), there are three resonant bands $\text{RB}_{\mathcal{L}_q} = \{B_1, B_2, B_3\}$. The standing waves $\nabla(B_i)$ s are computed as follows.

$$\begin{pmatrix} \nabla(B_1) \\ \nabla(B_2) \\ \nabla(B_3) \end{pmatrix} = \begin{pmatrix} \Delta(48) & \Delta(8) & \Delta(18) & \Delta(478) & \Delta(1478) & \Delta(178) \\ \Delta(3) & \Delta(34) & \Delta(134) & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta(3) & \Delta(13) & \Delta(134) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix},$$

where we use the abbreviation $\Delta(ijk) := q_{ijk}^{1/2} - q_{ijk}^{-1/2}$. It is easily seen that $\text{Ker } \nabla \neq 0$ if and only if the following six 2×2 minors are all zero.

$$\begin{aligned} D_1 &= \det \begin{pmatrix} \Delta(48) & \Delta(8) \\ \Delta(3) & \Delta(34) \end{pmatrix} = \Delta(4)\Delta(348), \\ D_2 &= \det \begin{pmatrix} \Delta(8) & \Delta(18) \\ \Delta(34) & \Delta(134) \end{pmatrix} = \pm \Delta(1)\Delta(128), \\ D_3 &= \det \begin{pmatrix} \Delta(48) & \Delta(18) \\ \Delta(3) & \Delta(134) \end{pmatrix} = \Delta(4)\Delta(1348) + q_{1234}^{1/2}\Delta(1)\Delta(1248), \\ D_4 &= \det \begin{pmatrix} \Delta(478) & \Delta(1478) \\ \Delta(3) & \Delta(13) \end{pmatrix} = \pm \Delta(1)\Delta(136), \\ D_5 &= \det \begin{pmatrix} \Delta(1478) & \Delta(178) \\ \Delta(13) & \Delta(134) \end{pmatrix} = \pm \Delta(4)\Delta(246), \\ D_6 &= \det \begin{pmatrix} \Delta(478) & \Delta(178) \\ \Delta(3) & \Delta(134) \end{pmatrix} = q_{1234}^{1/2}\Delta(4)\Delta(1246) + \Delta(1)\Delta(1346). \end{aligned}$$

Since 5678 and 235 are resonant points, $q_{1234} = q_{14678} = 1$. By the relations $D_1 = D_2 = D_4 = D_5 = 0$, it is natural to divide into four cases:

- (i) $\Delta(4) = \Delta(1) = 0$.
- (ii) $\Delta(4) \neq 0, \Delta(1) = \Delta(348) = \Delta(246) = 0$.
- (iii) $\Delta(1) \neq 0, \Delta(4) = \Delta(128) = \Delta(136) = 0$.
- (iv) $\Delta(4) \neq 0, \Delta(1) \neq 0, \Delta(348) = \Delta(136) = \Delta(128) = \Delta(246) = 0$.

The case (i) can not happen. Indeed, (i) implies that $q_6q_7q_8 = q_2q_3 = 1$ and then we have $q_5 = 1$, which contradicts the assumption.

(ii) implies that

$$q_1 = q_3q_4q_8 = q_2q_4q_6 = q_2q_3q_5 = q_1q_4q_6q_7q_8 = q_5q_6q_7q_8 = q_1q_2q_3q_4 = 1.$$

From these relations, we obtain $q_3 = q_6, q_4 = q_5, q_2 = q_8$ and $q_7 = 1$. These parameters form a component

$$C_{(28|36|45)} = \{(1, s_1, s_2, s_3, s_3, s_2, 1, s_1) \in (\mathbb{C}^*)^8 \mid s_1s_2s_3 = 1\} \quad (9)$$

corresponding to the braid subarrangement $\{H_2, H_3, H_4, H_5, H_6, H_8\}$. Similarly, from (iii), we have

$$C_{(15|26|38)} = \{(s_1, s_2, s_3, 1, s_1, s_2, 1, s_3) \in (\mathbb{C}^*)^8 \mid s_1s_2s_3 = 1\} \quad (10)$$

corresponding to the braid subarrangement $\{H_1, H_2, H_3, H_5, H_6, H_8\}$.

(iv) obviously implies that $D_1 = D_2 = D_4 = D_5 = 0$. Since $q_{348}^{1/2}, q_{136}^{1/2}, q_{128}^{1/2}, q_{246}^{1/2} \in \{\pm 1\}$, other conditions are equivalent to

$$\begin{aligned} D_3 = 0 &\iff q_{348}^{1/2} + q_{128}^{1/2} q_{1234}^{1/2} = 0, \\ D_6 = 0 &\iff q_{136}^{1/2} + q_{246}^{1/2} q_{1234}^{1/2} = 0. \end{aligned}$$

Since the choice of $q_i^{1/2}$ has freedom of the sign, we may assume that $q_{348}^{1/2} = q_{128}^{1/2} = 1$. Then we have $q_{34}^{1/2} = q_{12}^{1/2} = q_8^{-1/2}$. Then $D_3 = 0$ is equivalent to $q_{1234}^{1/2} = -1$, which implies that

$$\begin{aligned} D_3 = 0 &\iff q_{12}^{1/2} = q_{34}^{1/2} = q_8^{-1/2} = i \\ D_6 = 0 &\iff q_{13}^{1/2} = q_{24}^{1/2} (= \pm i). \end{aligned}$$

Set $q_1^{1/2} = \lambda$. Then $q_2^{1/2} = i\lambda^{-1}$, $q_3^{1/2} = \pm i\lambda^{-1}$, $q_4^{1/2} = \pm\lambda$, $q_5^{1/2} = \mp\lambda^2$, $q_6^{1/2} = \mp i q_{136}^{1/2}$, $q_7^{1/2} = \pm q_{136}^{1/2} \lambda^{-2}$, $q_8^{1/2} = -i$. The parameters $q_i = (q_i^{1/2})^2$ form the following component (we set $s = \lambda^2$)

$$\Omega = \{(s, -s^{-1}, -s^{-1}, s, s^2, -1, s^{-2}, -1) \in (\mathbb{C}^*)^8 \mid s \in \mathbb{C}^*\}.$$

It is the so-called translated component [6, 5].

4.3 $q_7 \neq 1$

This case is quite similar to the case $q_5 \neq 1$. If $H^1(\mathcal{L}_q) \neq 0$, then we have:

$$q \in C_{5678} \cup C_{147} \cup C_{(18|37|46)} \cup C_{(16|27|48)} \cup \Omega. \quad (11)$$

4.4 $q_5 = q_7 = 1, q_8 \neq 1$

In this case, we compute H^1 with Figure 4.

The set of resonant bands $\text{RB}_{\mathcal{L}}$ is a subset of $\{B_1, B_2, B_3, B_4\}$. The coefficients of $\nabla(B_i)$ are as follows.

	C_1	C_2	C_3	C_4	C_5	C_6
$\nabla(B_1)$	0	$\Delta(3)$	0	0	$\Delta(36)$	0
$\nabla(B_2)$	$\Delta(1)$	$\Delta(13)$	$\Delta(123)$	0	0	0
$\nabla(B_3)$	0	0	0	$\Delta(3)$	$\Delta(13)$	$\Delta(134)$
$\nabla(B_4)$	0	$\Delta(16)$	0	0	$\Delta(1)$	0

There are three multiple points 128, 5678 and 348 on H_8 . We divide into 8 cases.

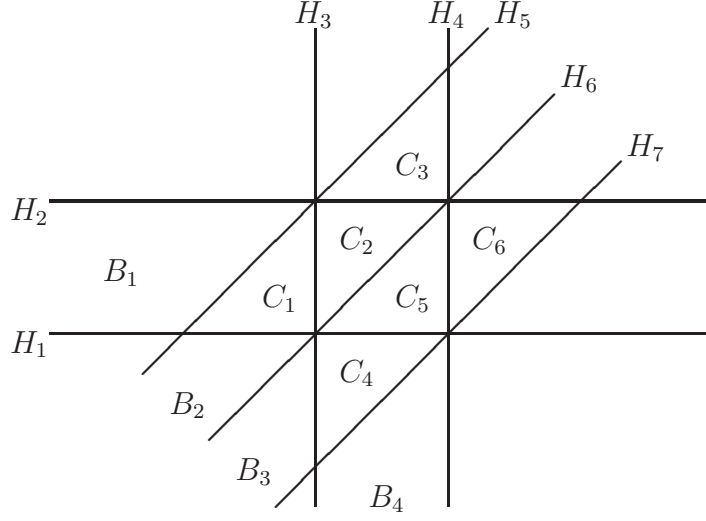


Figure 4: The deleted B_3 -arrangement with line at infinity H_8

- (i) $q_{128} \neq 1, q_{5678} \neq 1, q_{348} \neq 1$. By Corollary 3.8, $H^1 = 0$.
- (ii) $q_{128} = 1, q_{5678} \neq 1, q_{348} \neq 1$. By Theorem 3.11, $H^1 \neq 0$ if and only if $q_3 = q_4 = q_5 = q_6 = q_7 = 1$. Hence $\mathbf{q} = (q_1, \dots, q_8)$ is contained in C_{128} .
- (iii) $q_{128} \neq 1, q_{5678} = 1, q_{348} \neq 1$. By Theorem 3.11, $H^1 \neq 0$ if and only if $q_1 = q_2 = q_3 = q_4 = 1$. Hence $\mathbf{q} = (q_1, \dots, q_8)$ is contained in C_{5678} .
- (iv) $q_{128} = 1, q_{5678} = 1, q_{348} \neq 1$. If $H^1(\mathcal{L}_{\mathbf{q}}) \neq 0$, then we can prove that \mathbf{q} is contained in

$$\{(1, t^{-1}, t, 1, 1, t^{-1}, 1, t) \mid t \in \mathbb{C}^*\} \cup \{(t^{-1}, 1, 1, t, 1, t^{-1}, 1, t) \mid t \in \mathbb{C}^*\},$$

which is a subset of $C_{(15|26|38)} \cup C_{(16|27|48)}$.

- (v) $q_{128} \neq 1, q_{5678} \neq 1, q_{348} = 1$. If $H^1(\mathcal{L}_{\mathbf{q}}) \neq 0$, then $\mathbf{q} \in C_{348}$.
- (vi) $q_{128} = 1, q_{5678} \neq 1, q_{348} = 1$. In this case, there are two resonant bands B_1 and B_4 . Thus $H^1(\mathcal{L}_{\mathbf{q}}) \neq 0$ if and only if the minor

$$\det \begin{pmatrix} \Delta(3) & \Delta(36) \\ \Delta(16) & \Delta(1) \end{pmatrix} = -\Delta(6)\Delta(136)$$

is equal to zero. Suppose that $\Delta(6) = 0$. Then since $q_{34567} = q_5 = q_7 = 1$, we have $q_{34} = 1$, which implies $q_8 = 1$. Hence we may assume that

$\Delta(6) \neq 0$ and $\Delta(136) = 0$. We can conclude that $H^1 \neq 0$ if and only if \mathbf{q} is contained in

$$C_{(14|23|68)} = \{(s_1, s_2, s_2, s_1, 1, s_3, 1, s_3) \mid s_i \in \mathbb{C}^*, s_1 s_2 s_3 = 1\}.$$

(vii) $q_{128} \neq 1, q_{5678} = 1, q_{348} = 1$. It is similar to the case (iv). We can prove that if $H^1 \neq 0$, then \mathbf{q} is contained in

$$C_{(28|36|45)} \cup C_{(18|37|46)}.$$

(viii) $q_{128} = 1, q_{5678} = 1, q_{348} = 1$. We can prove that $\mathbf{q} \in C_{(14|23|68)}$.

4.5 $q_5 = q_7 = q_8 = 1, q_6 \neq 1$

We use Figure 5. This case is similar to the previous case. Two new components C_{136} and C_{246} appear. Other parameters are contained in components which have been already known.

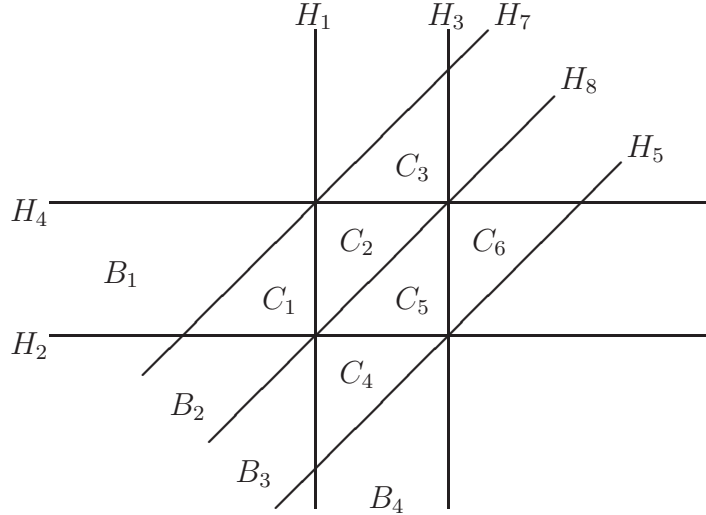


Figure 5: The deleted B_3 -arrangement with line at infinity H_6

4.6 $q_5 = q_6 = q_7 = q_8 = 1, q_3 \neq 1$

We use Figure 6. There are three bands B_1, B_2, B_3 .

	C_1	C_2	C_3	C_4	C_5	C_6
$\nabla(B_1)$	$\Delta(4)$	$\Delta(4)$	$\Delta(24)$	$\Delta(24)$	0	$\Delta(24)$
$\nabla(B_2)$	$\Delta(1)$	$\Delta(1)$	$\Delta(12)$	$\Delta(12)$	$\Delta(12)$	0
$\nabla(B_3)$	0	0	$\Delta(1)$	$\Delta(1)$	$\Delta(1)$	$\Delta(1)$

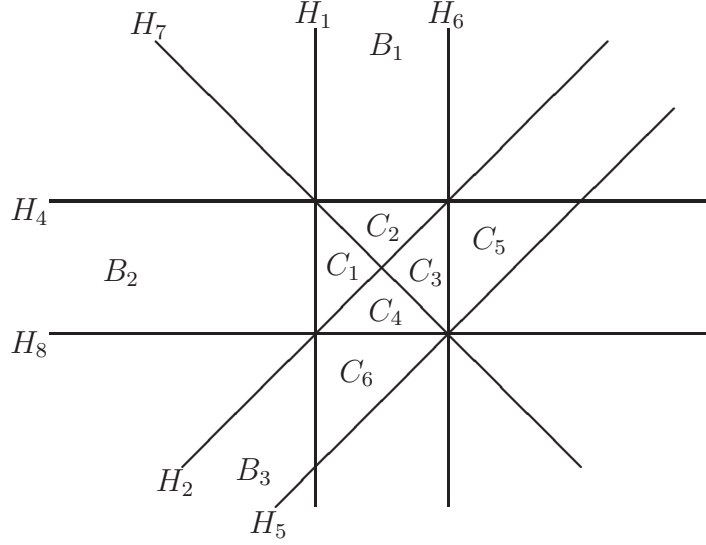


Figure 6: The deleted B_3 -arrangement with line at infinity H_3

If bands B_1 , B_2 and B_3 are all resonant, then $\Delta(13) = \Delta(24) = \Delta(12) = \Delta(34) = \Delta(14) = \Delta(23) = 0$. We have $q_1 = q_2 = q_3 = q_4 = -1$, and so $\mathbf{q} \in C_{(14|23|68)}$. Other cases are similar. Consequently, in this case, $H^1 \neq 0$ implies that

$$\mathbf{q} \in C_{235} \cup C_{348} \cup C_{(14|23|68)}.$$

So the parameter \mathbf{q} is contained in the known components.

4.7 $q_3 = q_5 = q_6 = q_7 = q_8 = 1$

The remaining case is easily seen that $H^1(\mathcal{L}_{\mathbf{q}}) = 0$ if $\mathbf{q} \neq (1, 1, \dots, 1)$.

Consequently, we have the decomposition of the characteristic variety as follows.

$$\begin{aligned} V_1(M(\mathcal{A})) = & C_{136} \cup C_{147} \cup C_{235} \cup C_{128} \cup C_{246} \cup C_{348} \cup C_{5678} \cup \\ & C_{(14|23|68)} \cup C_{(28|36|45)} \cup C_{(15|26|38)} \cup C_{(18|37|46)} \cup C_{(16|27|48)} \cup \Omega. \end{aligned}$$

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